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### An extension of König's theorem to graphs with no odd-K4

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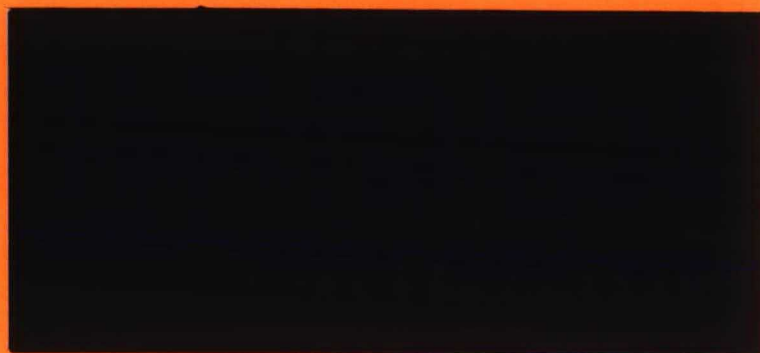
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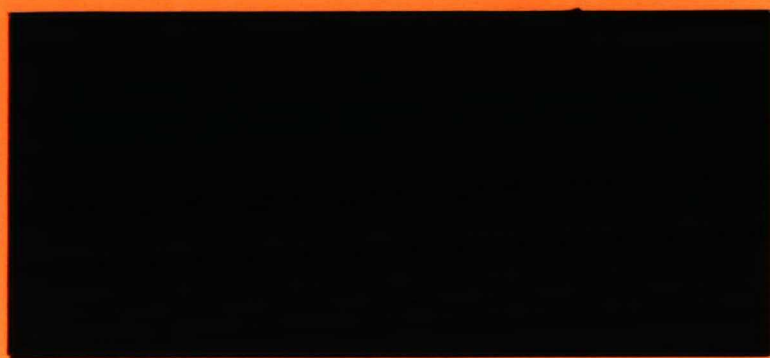
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## RESEARCH MEMORANDUM



TILBURG UNIVERSITY  
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AN EXTENSION OF KÖNIG'S THEOREM TO  
GRAPHS WITH NO ODD- $K_4$

by

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## Abstract

We prove the following min-max relations. Let  $G$  be an undirected graph, without isolated nodes, not containing an odd- $K_4$  (a homeomorph of  $K_4$  (the 4-clique) in which the triangles of  $K_4$  have become odd cycles). Then the maximum cardinality of a stable set in  $G$  is equal to the minimum cost of a collection of edges and odd circuits in  $G$ , covering the nodes of  $G$ . Here the cost of an edge is 1 and the cost of a circuit of length  $2k+1$  equal to  $k$ .

Moreover, the minimum cardinality of a node-cover for  $G$  is equal to the maximum profit of a collection mutually node disjoint edges and odd circuits in  $G$ . Here the profit of an edge is 1 and the profit of a circuit of length  $2k+1$  is equal to  $k+1$ . Also weighted versions of these min-max relations hold. The result extends König's well-known min-max relations for stable sets and node-covers in bipartite graphs. Moreover it extends results of Chvátal, Boulala, Fonlupt, and Uhry. A weaker, fractional, version of these min-max relations follows from earlier results obtained by Schrijver and the author.

## 1. Introduction

The subject of this paper is to give an extension of the following well-known result.

(1.1) If  $G$  has no odd circuit,  
       then  $\alpha(G) = \rho(G)$  and  $\tau(G) = \nu(G)$  (König [1931,1933])

Here, and in the sequel,  $G = (V(G), E(G))$  denotes an undirected graph without isolated nodes. As usual, the parameters  $\alpha$ ,  $\rho$ ,  $\tau$  and  $\nu$  are defined by:

$\alpha(G)$  = the maximum cardinality of a stable set in  $G$ . ( $S \subset V(G)$  is a stable set if  $u, v \in S$  implies  $uv \notin E(G)$ .)

$\rho(G)$  = the minimum cardinality of an edge-cover for  $G$ . ( $E' \subset E(G)$  is an edge-cover if for each  $u \in V$  there exists an  $e \in E'$  covering  $u$ .)

$\nu(G)$  = the maximum cardinality of a matching in  $G$ . ( $M \subset E(G)$  is a matching if  $e_1, e_2 \in M$ ,  $e_1 \neq e_2$  implies  $e_1 \cap e_2 = \emptyset$ .)

$\tau(G)$  = the minimum cardinality of a node-cover for  $G$ . ( $N \subset V(G)$  is a node-cover if  $uv \in E(G)$  implies  $u \in N$  or  $v \in N$ .)

We introduce two new parameters:

$\tilde{\rho}(G)$  = the minimum cost of a collection of edges and odd circuits in  $G$  covering the nodes of  $G$ . The cost of an edge is equal to 1, and the cost of a circuit with  $2k+1$  edges is equal to  $k$ . The cost of a collection of edges and odd circuits is equal to the sum of the costs of its members.

$\tilde{\nu}(G)$  = the maximum profit of a collection of mutually node disjoint edges and odd circuits in  $G$ . The profit of an edge is equal to 1 and the profit of a circuit of length  $2k+1$  is equal to  $k+1$ . The

profit of a collection of edges and odd circuits is equal to the sum of the profits of its members.

The following inequalities are obvious:

$$\alpha(G) \leq \tilde{\rho}(G) \leq \rho(G),$$

(1.2)

$$\tau(G) \geq \tilde{v}(G) \geq v(G).$$

König's Theorem (1.1) can be extended to the following result. (It follows from the more general Theorem 1.8, which will be proved in section 2.)

### Theorem 1.3

Let  $G$  be an undirected graph, without isolated nodes. If  $G$  does not contain any  $\text{odd-}K_4$  as a subgraph, then  $\alpha(G) = \tilde{\rho}(G)$  and  $\tau(G) = \tilde{v}(G)$ .  $\square$

An  $\text{odd-}K_4$  is a homeomorph of  $K_4$  (the 4-clique) in which all triangles have become odd circuits. (See figure 1, wriggled lines stand for pairwise openly disjoint paths; odd indicates that the corresponding faces are odd circuits.)

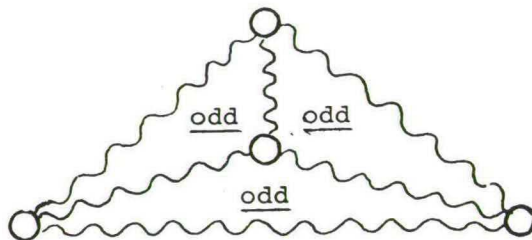


figure 1

To see that Theorem 1.3 extends König's Theorem (1.1), observe that a bipartite graph  $G$  has no  $\text{odd-}K_4$ , and satisfies  $\tilde{\rho}(G) = \rho(G)$ ,  $\tilde{\tau}(G) = \tau(G)$  (as  $G$  has no odd circuits.)

The two equalities in (1.1) are equivalent, for any graph  $G$ . This follows from

$$(1.4) \quad \alpha(G) + \tau(G) = |V(G)| = \rho(G) + \nu(G) \quad (\text{Gallai [1958,1959]}).$$

A similar equivalence for the equalities  $\alpha(G) = \tilde{\rho}(G)$  and  $\tau(G) = \tilde{\nu}(G)$  follows from the following result observed by Schrijver, analogous to Gallai's result above.

Theorem 1.5

Let  $G$  be an undirected graph without isolated nodes. Then  $\tilde{\rho}(G) + \tilde{\nu}(G) = |V(G)|$ .

Proof:

First, let  $e_1, \dots, e_m, C_1, \dots, C_n$  be a collection of mutually node disjoint edges and odd circuits such that the profit  $m + \sum_{i=1}^n \frac{1}{2}(|V(C_i)| + 1)$  of the collection is equal to  $\tilde{\nu}(G)$ .

Let  $V_1 := V(G) \setminus \bigcup_{i=1}^n V(C_i)$ , and let  $G_1$  be the subgraph of  $G$  induced by  $V_1$ . Then obviously  $m = \nu(G_1)$ . Let  $f_1, \dots, f_{\rho(G_1)}$  be a minimum edge cover for  $G_1$ . Then  $f_1, \dots, f_{\rho(G_1)}, C_1, \dots, C_n$  is a collection of edges and odd circuits covering  $V(G)$ . The cost of this collection is (using Gallai's identity (1.4)):

$$\begin{aligned} \rho(G_1) + \sum_{i=1}^n \frac{1}{2}(|V(C_i)| - 1) &= |V_1| - \nu(G_1) - \sum_{i=1}^n \frac{1}{2}(|V(C_i)| + 1) + \sum_{i=1}^n |V(C_i)| \\ &= |V(G)| - \tilde{\nu}(G). \end{aligned}$$

Hence  $\tilde{\rho}(G) + \tilde{\nu}(G) \leq |V(G)|$ .

The reverse inequality is proved almost identically. However there is a small technical difference, dealt with in the claim below.

Let  $e_1, \dots, e_m, C_1, \dots, C_n$  be a collection of edges and odd circuits covering  $V(G)$  such that the cost  $m + \sum_{i=1}^n \frac{1}{2}(|V(C_i)| - 1)$  of the collection is equal to  $\tilde{\rho}(G)$ , and  $n$  is as small as possible.

Claim: For each  $i, j=1, \dots, n$  ( $i \neq j$ );  $k=1, \dots, m$  we have  $V(C_i) \cap V(C_j) = \emptyset$ ,  $V(C_i) \cap e_k = \emptyset$ .

Proof of Claim: Suppose,  $u \in V(C_i)$  ( $i=1, \dots, n$ ), such that  $u$  is also contained in another odd circuit among  $C_1, \dots, C_n$ , or in one of the edges  $e_1, \dots, e_m$ . Let  $f_1, \dots, f_p \in E(C_i)$  be the unique maximum cardinality matching in  $C_i$  not covering  $u$ . Then  $p = \frac{1}{2}(|V(C_i)| - 1)$ . Obviously  $e_1, \dots, e_m, f_1, \dots, f_p, C_1, \dots, C_{i-1}, C_{i+1}, \dots, C_n$  is a collection of edges and odd circuits covering  $V(G)$ . Its cost is  $\tilde{\rho}(G)$ . However it contains only  $n-1$  odd circuits, contradicting the minimality of  $n$ .

end of proof of claim.

As before we define  $V_1 = V(G) \setminus \bigcup_{i=1}^n V(C_i)$  and  $G_1$  as the subgraph of  $G$  induced by  $V_1$ . By similar arguments as used in the first part of the proof one gets:

$$\begin{aligned} \tilde{\rho}(G) &= \rho(G_1) + \sum_{i=1}^n \frac{1}{2}(|V(C_i)| - 1) = |V_1| - v(G_1) - \sum_{i=1}^n \frac{1}{2}(|V(C_i)| + 1) \\ &+ \sum_{i=1}^n |V(C_i)| \geq |V(G)| - \tilde{v}(G). \end{aligned} \quad \square$$

#### Corollary 1.6

Let  $G$  be an undirected graph without isolated nodes. Then  $\alpha(G) = \tilde{\rho}(G)$  if and only if  $\tau(G) = \tilde{v}(G)$ .  $\square$

As mentioned, we prove Theorem 1.3 in section 2. In fact we shall prove a more general weighted version of this theorem (Theorem 1.8 below).

#### Weighted versions

We define weighted versions of the numbers  $\alpha$ ,  $\rho$ ,  $v$ ,  $\tau$ ,  $\tilde{\rho}$ , and  $\tilde{v}$  and state the obvious generalizations of the results mentioned.

Let  $w \in \mathbb{Z}^{V(G)}$ .

$$\alpha_w(G) = \text{maximum } \left\{ \sum_{u \in S} w_u \mid S \text{ is a stable set in } G \right\},$$

$\rho_w(G)$  = the minimum cardinality of a  $w$ -edge-cover for  $G$ . (A  $w$ -edge-cover for  $G$  is a collection  $e_1, \dots, e_m$  in  $E(G)$  (repetition allowed) such that for each  $u \in V(G)$  there are at least  $w_u$  edges among  $e_1, \dots, e_m$  incident with  $u$ . The cardinality of  $e_1, \dots, e_m$  is  $m$ .)

$\nu_w(G)$  = the maximum cardinality of a  $w$ -matching in  $G$ . (A  $w$ -matching is a collection  $e_1, \dots, e_m$  in  $E(G)$  (repetition allowed) such that for each  $u \in E(G)$  there are at most  $w_u$  edges among  $e_1, \dots, e_m$  incident with  $u$ .)

$\tau_w(G)$  = minimum  $\{ \sum_{u \in N} w_u \mid N \text{ is node-cover for } G \}$ .

Moreover we define:

A  $w$ -cover ( $w$ -packing respectively) by edges and odd circuits is a collection  $e_1, \dots, e_m$  of edges and  $C_1, \dots, C_m$  of odd circuits (repetition allowed), such that for each  $u \in V(G)$ :

$|\{i=1, \dots, m \mid u \text{ incident with } e_i\}| + |\{i=1, \dots, n \mid u \in V(C_i)\}| \geq w_u (\leq w_u \text{ respectively}).$

The cost of  $e_1, \dots, e_m, C_1, \dots, C_n$  is  $m + \sum_{i=1}^n \frac{1}{2}(|V(C_i)| - 1)$ , its profit is  $m + \sum_{i=1}^n \frac{1}{2}(|V(C_i)| + 1)$ .

$\tilde{\rho}_w(G)$  = the minimum cost of a  $w$ -cover by edges and odd circuits for  $G$ .

$\tilde{\nu}_w(G)$  = the maximum profit of a  $w$ -packing by edges and odd circuits in  $G$ .

These numbers satisfy:

If  $G$  has no odd circuit, then  $\alpha_w(G) = \rho_w(G)$  and  $\tau_w(G) = \nu_w(G)$   
(Egerváry [1931]),

$$(1.7) \quad \begin{aligned} \alpha_w(G) &\leq \tilde{\rho}_w(G) \leq \rho_w(G), \\ \tau_w(G) &\geq \tilde{\nu}_w(G) \geq \nu_w(G), \\ \alpha_w(G) + \tau_w(G) &= \tilde{\rho}_w(G) + \tilde{\nu}_w(G) = \rho_w(G) + \nu_w(G) = \sum_{u \in V(G)} w_u. \end{aligned}$$

(1.7) can be proved easily from the cardinality versions stated before (with  $w \equiv 1$ ), using the following construction:



Define  $G_w$  by:

$$V(G_w) = \{[u, i] \mid u \in V(G); i=1, \dots, w_u\},$$

$$E(G_w) = \{[u, i][v, j] \mid u, v \in V(G); uv \in E(G); i=1, \dots, w_u; j=1, \dots, w_v\}.$$

Then one easily proves that  $\alpha_w(G) = \alpha(G_w)$ ,  $\rho_w(G) = \rho(G_w)$ ,  $\nu_w(G) = \nu(G_w)$ ,  $\tau_w(G) = \tau(G_w)$ ,  $\tilde{\rho}_w(G) = \tilde{\rho}(G_w)$ ,  $\tilde{\nu}_w(G) = \tilde{\nu}(G_w)$ , and  $V(G_w) = \sum_{u \in V(G)} w_u$ .

Moreover  $G_w$  is bipartite if and only if  $G$  is. These yield (1.7). Theorem 1.3 can be generalized as well.

### Theorem 1.8

Let  $G$  be an undirected graph, without isolated nodes. If  $G$  does not contain any odd- $K_4$  as a subgraph, then  $\alpha_w(G) = \tilde{\rho}_w(G)$  and  $\tau_w(G) = \tilde{\nu}_w(G)$  for any  $w \in \mathbb{Z}^{V(G)}$ . □

The proof of Theorem 1.8 is in section 2. It should be noted that Theorem 1.8 does not follow from Theorem 1.3 by using  $G_w$ . The reason is that it is possible that  $G_w$  contains an odd- $K_4$  even if  $G$  does not. This is illustrated by the graph in figure 2. (The bold edges, in figure 2b form an odd- $K_4$ .)

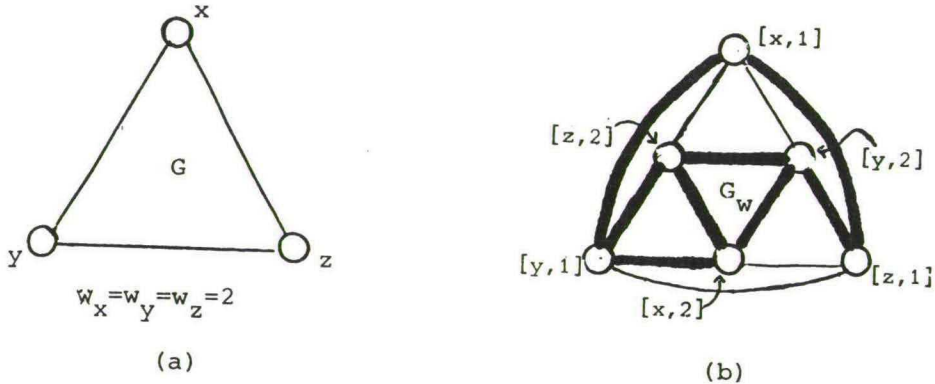


figure 2

The statement " $\alpha_w(G) = \tilde{\rho}_w(G)$  for each  $w \in \mathbb{Z}^{V(G)}$ " can be reformulated in terms of integer linear programming.

(1.9) Both optima in the following primal-dual pair of linear programs, are attained by integral vector if  $w$  is integer valued.

Primal:

$$\begin{aligned} \tilde{\rho}_w^*(G) &:= \max \sum_{u \in V(G)} w_u x_u \\ \text{s.t. } x_u + x_v &\leq 1 & (uv \in E(G)), \\ \sum_{u \in V(C)} x_u &\leq \frac{1}{2}(|V(C)| - 1) & (C \in \mathcal{C}(G)), \\ x_u &\geq 0 & (u \in V(G)). \end{aligned}$$

Dual:

$$\begin{aligned} \tilde{\rho}_w^*(G) &= \min \sum_{e \in E(G)} y_e + \sum_{C \in \mathcal{C}(G)} \frac{1}{2}(|V(C)| - 1) z_C \\ \text{s.t. } \sum_{\substack{e \in E(G) \\ e \ni u}} y_e + \sum_{\substack{C \in \mathcal{C}(G) \\ V(C) \ni u}} z_C &\geq w_u & (u \in V(G)), \\ y_e &\geq 0 & (e \in E(G)), \\ z_C &\geq 0 & (C \in \mathcal{C}(G)). \end{aligned}$$

( $\mathcal{C}(G)$  denotes the collection of odd circuits  $C = (V(C), E(C))$  in  $G$ .)

So Theorem (1.8) implies that if  $G$  has no odd- $K_4$ , then  $\tilde{\rho}_w(G) = \tilde{\rho}_w^*(G)$  for each  $w \in \mathbb{Z}_+^{V(G)}$ . In other words, the system of linear inequalities in the primal problem of (1.9) is totally dual integral (cf. Edmonds-Giles [1977]). Consequently (Edmonds-Giles [1977], Hoffman [1974]), if  $G$  has no odd- $K_4$ , then  $\alpha_w(G) = \tilde{\rho}_w^*(G)$  for each  $w \in \mathbb{Z}_+^{V(G)}$ . This means that the system of linear inequalities in the primal problem of (1.9) describes the stable set polytope of  $G$ . (The stable set polytope of  $G$  is the convex hull of the characteristic vectors of the stable sets of  $G$ , considered as subsets of  $V(G)$ .)

Obviously, also the statement " $\tau_w(G) = \tilde{v}_w(G)$  for each  $w \in \mathbb{Z}_+^{V(G)}$ ," can be formulated in a way similar to (1.9).

We conclude this section with some remarks. Section 2 contains the proof of Theorem 1.3 and 1.8. Finally, in section 3, we consider some algorithmic aspects of the results in this paper.

### Remarks

(i) Earlier results on this topic are:

- Chvátal [1975]: If  $G$  is series-parallel (i.e.  $G$  contains no homomorph of  $K_4$ ), then  $\alpha(G) = \tilde{\rho}(G)$ .

- Boulala and Uhry [1979]: If  $G$  is series-parallel, then  $\alpha_w(G) = \tilde{\rho}_w(G)$  for each  $w \in \mathbb{Z}^{V(G)}$ . (In fact they only emphasize  $\alpha_w(G) = \tilde{\rho}_w^*(G)$  (which was conjectured by Chvátal [1975])). But their proof implicitly yields the stronger result. Recently Mahjoub [1985] gave a very short proof of  $\alpha_w(G) = \tilde{\rho}_w^*(G)$  for each  $w \in \mathbb{Z}^{V(G)}$  for series-parallel graphs  $G$ .)

- Fonlupt and Uhry [1982]: If there exists a  $u \in V(G)$  such that  $u \in V(C)$  for all  $C \in \mathcal{C}(G)$ , then  $\alpha_w(G) = \tilde{\rho}_w^*(G)$  for each  $w \in \mathbb{Z}^{V(G)}$ . Sbihi and Uhry [1984] give a new proof of Fonlupt and Uhry's result. This proof implicitly yields  $\alpha_w(G) = \tilde{\rho}_w(G)$  for each  $w \in \mathbb{Z}^{V(G)}$ .

Obviously, the graphs considered by Chvátal, Boulala, Fonlupt, Sbihi, and Uhry do not contain an odd- $K_4$ .

- Gerards and Schrijver [1985]: If  $G$  has no odd- $K_4$  then  $\alpha_w(G) = \tilde{\rho}_w^*(G)$  for each  $w \in \mathbb{Z}^{V(G)}$ .

(ii) Theorem 1.8 (and 1.3) can be refined by allowing  $w$ -covers ( $w$ -packings) by edges and odd circuits only to use edges not contained in a triangle, and odd circuits not having a chord. In other words, if  $G$  has no odd- $K_4$ , then the system:

$$x_u + x_v \leq 1 \quad (uv \in E(G), uv \text{ is not contained in a triangle})$$

$$(*) \quad \sum_{u \in V(C)} x_u \leq \frac{1}{2}(|V(C)| - 1) \quad (C \in \mathcal{C}(G), C \text{ has no chord})$$

$$x_u \geq 0 \quad (u \in V(G))$$

is a totally dual integral system defining the stable set polytope of  $G$ . In fact the inequalities in (\*) are all facets of the polyhedron defined by (\*) (for any graph  $G$ ). So (\*) is the unique minimal totally dual integral system (cf. Schrijver [1981]) for the stable set polytope of  $G$ , in case  $G$  has no odd- $K_4$ .

(iii) Lovász, Schrijver, Seymour, and Truemper [1984] give a constructive characterization of graphs with no  $\text{odd-}K_4$ :  $G$  has no  $\text{odd-}K_4$  if and only if one of the following holds:

- There exists a  $u \in V(G)$  such that  $u \in V(C)$  for all  $C \in \mathcal{C}(G)$  (Fonlupt and Uhry's case mentioned in remark (i) above).
- $G$  is planar, and at most two faces of  $G$  are odd circuits.
- $G$  is the graph in figure 3.
- $G$  can be decomposed into smaller graphs with no  $\text{odd-}K_4$ .

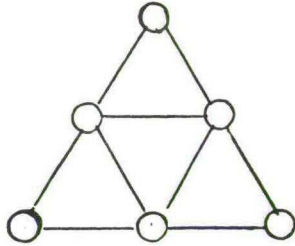


figure 3

## 2. The proof of Theorem 1.8

We first derive a special case of Theorem 1.8. To state and prove it we need some extra notions and an auxiliary result (Theorem 2.1). An  $\text{odd-}K_3^2$  is a graph as indicated in figure 4 (wiggles and dotted lines stand for pairwise openly disjoint paths, dotted lines may have length zero, wiggles lines have always positive length, odd indicates that the corresponding faces are odd cycles).

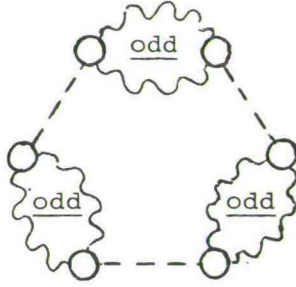


figure 4

An orientation of an undirected graph  $G$  is a directed graph obtained from  $G$  by directing the edges. We say that a directed graph has discrepancy 1 if in each circuit the number of forwardly directed arcs minus the number of backwardly directed arcs is 0 or  $\pm 1$ .

Theorem 2.1 (Gerards and Schrijver [1986])

Let  $G$  be an undirected graph. Then  $G$  does not contain an  $\text{odd-}K_4$  or an  $\text{odd-}K_3^2$  if and only if  $G$  has an orientation with discrepancy 1.  $\square$

Using this theorem we obtain the following special case of Theorem 1.8.

Theorem 2.2

Let  $G$  be an undirected graph without isolated nodes. If  $G$  does not contain any  $\text{odd-}K_4$  or any  $\text{odd-}K_3^2$ , then  $\alpha_w(G) = \tilde{\rho}_w(G)$  and  $\tau_w(G) = \tilde{v}_w(G)$  for each  $w \in \mathbb{Z}^{\tilde{V}(G)}$ .

Proof:

According to Theorem 2.1,  $G$  has an orientation with discrepancy 1. Let  $\vec{A}$  denote the set of arcs in this orientation. For each  $\vec{uv} \in \vec{A}$  we add a reversely directed arc  $\vec{vu}$  too. Denote  $\hat{A} := \{\vec{vu} | \vec{uv} \in \vec{A}\}$ .

Consider the following "circulation" problem:

$$\begin{aligned}
(2.3) \quad & \min \sum_{a \in \vec{A}} f_a \\
& \text{s.t.} \quad \sum_{\substack{a \in \vec{A} \cup \overleftarrow{A} \\ a \text{ enters } u}} f_a - \sum_{\substack{a \in \vec{A} \cup \overleftarrow{A} \\ a \text{ leaves } u}} f_a = 0 & (u \in V(G)) \\
& \quad \sum_{\substack{a \in \vec{A} \cup \overleftarrow{A} \\ a \text{ enters } u}} f_a \geq w_u & (u \in V(G)) \\
& \quad f_a \geq 0 & (a \in \vec{A} \cup \overleftarrow{A}),
\end{aligned}$$

and its linear programming dual:

$$\begin{aligned}
(2.4) \quad & \max \sum_{u \in V(G)} w_u x_u \\
& \text{s.t.} \quad \pi_v - \pi_u + x_v \leq 1 & (\overrightarrow{uv} \in \vec{A}) \\
& \quad \pi_u - \pi_v + x_u \leq 0 & (\overrightarrow{vu} \in \vec{A}) \\
& \quad x_u \geq 0 & (v \in V(G)).
\end{aligned}$$

The theorem is proved by the following three propositions:

Proposition 1: The constraint matrix of (2.3) is totally unimodular. Consequently both (2.3) and (2.4) have integral optimal solutions (Hoffman and Kruskal [1956]).

Proposition 2: Let  $\pi \in \mathbb{Z}^{V(G)}$ ,  $x \in \mathbb{Z}^{V(G)}$  be a feasible solution of (2.4). Then  $x$  is a feasible solution of the primal problem of (1.9).

Proposition 3: Let  $f \in \mathbb{Z}^{\vec{A} \cup \overleftarrow{A}}$  be a feasible solution of (2.3). Then there exists a  $y \in \mathbb{Z}^{E(G)}$  and a  $z \in \mathbb{Z}^{C(G)}$ , which form a feasible solution of the dual problem of (1.9), such that:

$$\sum_{e \in E(G)} y_e + \sum_{C \in C(G)} \frac{1}{2}(|V(C)| - 1)z_C \leq \sum_{a \in \vec{A}} f_a.$$

Indeed, the three propositions together prove that  $\alpha_w(G) > \tilde{\rho}_w(G)$ . By (1.7), this yields  $\alpha_w(G) = \tilde{\rho}_w(G)$  and  $\tau_w(G) = \tilde{v}_w(G)$ .

The three propositions above are shown as follows:



Proof of Proposition 1:

If we are given a directed graph  $D = (V(D), A(D))$  and a spanning directed tree  $T = (V(D), T(D))$  on the same node set (not necessarily  $T(D) \subset A(D)$ ), then the network matrix  $N$  of  $D$  with respect to  $T$  is defined as follows:

$N \in \{0, 1, -1\}^{A(T) \times A(D)}$ . For  $u, v \in V(D)$  let  $P(u, v) \subset A(T)$  be the unique path in  $T$  from  $u$  to  $v$ . Then for each  $a_1 \in A(T)$ ,  $a_2 = \overrightarrow{uv} \in A(D)$ :

$$N_{a_1, a_2} := \begin{cases} 1 & \text{if } a_1 \in P(u, v), \text{ and } a_1 \text{ is passed forwardly going along} \\ & P(u, v) \text{ from } u \text{ to } v \\ -1 & \text{if } a_1 \in P(u, v), \text{ and } a_1 \text{ is passed backwardly going along} \\ & P(u, v) \text{ from } u \text{ to } v \\ 0 & \text{if } a_1 \notin P(u, v). \end{cases}$$

Network matrices are totally unimodular (Tutte [1965]). We prove Proposition 1 by proving that the constraint matrix of (2.3) is a network matrix.

Indeed, let  $V(D) := V(T) := \{v_0\} \cup \{[u, i] \mid u \in V(G), i \in \{1, 2\}\}$ ,  
 $A(D) := \{[\overrightarrow{u, 1}][\overrightarrow{v, 2}] \mid \overrightarrow{uv} \in \vec{A}\}$ , and  
 $A(T) := \{\overrightarrow{v_0[u, 1]} \mid u \in V(G)\} \cup \{\overrightarrow{u_1 u_2} \mid u \in V(G)\}$ .

Proof of Proposition 2:

Since  $x$  is integral we only need to prove that  $x_u + x_v \leq 1$  for  $uv \in E(G)$ . Indeed  $x_v + x_u \leq (1 - \pi_v + \pi_u) + (\pi_v - \pi_u) = 1$  if  $uv \in E(G)$  ( $\overrightarrow{uv} \in \vec{A}$ ).

Proof of Proposition 3:

We can write  $f$  as  $f = \sum_{D \in \Delta} \lambda_D f^D$ , where  $\Delta$  is a collection of directed circuits in  $\vec{A} \cup \vec{A}$ ,  $f^D \in \{0, 1\}^{\vec{A} \cup \vec{A}}$  with  $f_a^D = 1$  if and only if  $a \in D$ , and

$\lambda_D \in \mathbb{Z}_+$  for each  $D \in \Delta$ .

For every even circuit  $D \in \Delta$ , let  $M_D$  be an arbitrary maximum cardinality matching in  $\{uv \in E(G) \mid \overrightarrow{uv} \in D \text{ or } \overrightarrow{vu} \in D\}$ . (In particular if

$D = \{\overrightarrow{uv}, \overrightarrow{vu}\}$ . then  $M_D = \{uv\}$ .) Define  $y^D \in \mathbb{Z}^{E(G)}$  by:

$$y_e^D = \begin{cases} \lambda_D & \text{if } e \in M_D \\ 0 & \text{else.} \end{cases}$$

Next  $y \in \mathbb{Z}^{E(G)}$  is defined by  $y = \sum_{\substack{D \in \Delta \\ D \text{ even}}} y^D$ .

For each odd circuit  $D \in \Delta$ , let  $C_D \in \mathcal{C}(G)$  be defined by  $C_D = \{uv | \overrightarrow{uv} \in D \text{ or } \overrightarrow{vu} \in D\}$ . Define  $z \in \mathbb{Z}^{\mathcal{C}(G)}$  by:

$$z_C = \begin{cases} \lambda_D & \text{if } C = C_D \text{ for some } D, D \in \Delta, |D| \text{ odd} \\ 0 & \text{else.} \end{cases}$$

The vectors  $y \in \mathbb{Z}^{E(G)}$  and  $z \in \mathbb{Z}^{\mathcal{C}(G)}$  form a feasible solution to the dual problem of (1.9). Moreover

$$\begin{aligned} \sum_{a \in \vec{A}} f_a &= \sum_{D \in \Delta} \lambda_D |\vec{A} \cap D| \\ &\geq \sum_{\substack{D \in \Delta \\ D \text{ even}}} \lambda_D |M_D| + \sum_{\substack{D \in \Delta \\ D \text{ odd}}} \lambda_D \cdot \frac{1}{2} (|V(C_D)| - 1) \\ &= \sum_{e \in E(G)} y_e + \sum_{C \in \mathcal{C}(G)} \frac{1}{2} (|V(C)| - 1) z_C. \end{aligned} \quad \square$$

Before we prove Theorem 1.8 we state a result of Lovász and Schrijver [1984] (cf. Gerards-Schrijver, [1986, Theorem 2.6]). This result indicates that, in a sense, Theorem 2.2 is the core of Theorem 1.8.

### Theorem 2.5

Let  $G$  be an undirected graph, containing no odd- $K_4$ . If  $G$  contains an odd- $K_3^2$ , then one of the following holds

- (i)  $G$  is disconnected or has a one node cutset
- (ii)  $G$  has a two node cutset. Both sides of the cutset are not bipartite.  $\square$

Using this we finally prove Theorem 1.8.

### Proof of Theorem 1.8

Let  $G$  be a graph with no odd- $K_4$ . Assume that all graphs  $G'$  with  $|E(G')| < |E(G)|$  satisfy Theorem 1.8. We shall prove that then  $G$  satis-

fies Theorem 1.8. Obviously we may assume  $G$  to be connected. Let  $w \in \mathbb{Z}^{V(G)}$ . By the weighted version of Theorem 1.5 we only need to prove that  $\alpha_w(G) = \tilde{\rho}_w(G)$ . Obviously we may assume that  $w_u \geq 0$  for each  $u \in V(G)$ . According to Theorem 2.2 and 2.5 we may assume that  $G$  satisfies (i) or

(ii) of Theorem 2.5. So we have subsets  $V_1, V_2$  of  $V(G)$  such that  $|V_1 \cap V_2| \leq 2$ ,  $V_1 \cup V_2 = V(G)$ , and both  $V_1 \setminus V_2$  and  $V_2 \setminus V_1$  are non empty sets not joined by an edge in  $E(G)$ . Moreover, in case  $|V_1 \cap V_2| = 2$ , the subgraphs  $G_1$  and  $G_2$  in  $G$  induced by  $V_1, V_2$  respectively are not bipartite. In the sequel we shall use the following notation: For each stable set  $U \subset V_1 \cap V_2$  the number  $s(U)$  ( $s^1(U)$ ,  $s^2(U)$  respectively) denotes the maximum weight  $\sum_{u \in S} w_u$  of a stable set in  $G$  ( $G_1, G_2$  respectively) satisfying  $S \cap V_1 \cap V_2 = U$ . Note that:  $s(U) = s^1(U) + s^2(U) - \sum_{u \in U} w_u$  for each stable set  $U$  in  $V_1 \cap V_2$ . We consider two cases.

Case I:  $V_1 \cap V_2$  induces a clique in  $G$ .

Define the following weight functions:

$$w_u^1 := \begin{cases} w_u & \text{if } u \in V_1 \setminus V_2 \\ w_u + s^1(\emptyset) - s^1(\{u\}) & \text{if } u \in V_1 \cap V_2; \end{cases}$$

$$w_u^2 := \begin{cases} w_u & \text{if } u \in V_2 \setminus V_1 \\ s^1(\{u\}) - s^1(\emptyset) & \text{if } u \in V_1 \cap V_2. \end{cases}$$

Obviously  $G_1$  and  $G_2$  do not contain an odd- $K_4$ . Moreover  $|E(G_1)| < |E(G)|$ ,  $|E(G_2)| < |E(G)|$ . Hence there exist a  $w^1$ - and a  $w^2$ -cover by edges and odd circuits in  $G_1, G_2$  respectively, with cost  $s^1(\emptyset)$ ,  $\alpha_w(G) - s^2(\emptyset)$  respectively. The union of these two covers is a  $w$ -cover with edges and odd circuits in  $G$  with cost  $\alpha_w(G)$ . Hence  $\alpha_w(G) = \tilde{\rho}_w(G)$ .

Case II:  $|V_1 \cap V_2| = 2$ ,  $V_1 \cap V_2 = \{u_1, u_2\}$  say, and  $u_1 u_2 \notin E(G)$ . Define for  $i=1,2$ ;  $k=2,3$  the graph  $G_1^k$  by adding to  $G_1$  a path from  $u_1$  to  $u_2$  with  $k$ -edges. (See figures 5 and 6.)

Claim 1: We may assume that  $G_1^k$  does not contain an odd- $K_4$  ( $i=1,2$ ;  $k=2,3$ ). Moreover  $|E(G_1^k)| < |E(G)|$ .

Proof of Claim 1: To prove the first assertion (for  $i=1$ ), it is sufficient to prove that in  $G_2$  there exists an odd as well as an even path from  $u_1$  to  $u_2$ . Suppose this is not the case. Since  $G_2$  is not bipartite this implies the existence of a cutnode in  $G_2$  separating  $\{u_1, u_2\}$  from an odd cycle in  $G_2$ . But such a cutnode is also a cutnode of  $G$ . In that case we can apply Case I to prove  $\alpha_w(G) = \tilde{\rho}_w(G)$ . So we may assume that  $G_1^k$  has no odd- $K_4$ . If  $|E(G_1^k)| \geq |E(G)|$ , then  $|E(G_2)| \leq 3$ . Hence, since  $G_2$  is not bipartite,  $G_2$  is a triangle. So  $u_1 u_2 \in E(G)$ , contradicting our assumption that  $u_1 u_2 \notin E(G)$ . end of proof of claim 1

Define  $\Delta := s^2(\{u_1\}) + s^2(\{u_2\}) - s^2(\{u_1, u_2\}) - s^2(\emptyset)$ . Again we consider two cases.

Case IIa:  $\Delta \geq 0$ .

Let  $b_1, b_2$  be the new nodes in  $G_1^3$ ,  $b$  the new node in  $G_2^2$ . (See figure 5 below.) Moreover, let  $e_1, e_2, \tilde{e}, f_1$ , and  $f_2$  be the edges indicated in figure 5.

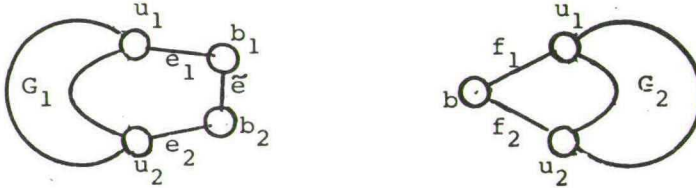


figure 5

We define the following weight functions:

$$w^1 \in \mathbb{Z}^{V(G_1^3)} \text{ by } w_u^1 := \begin{cases} w_u & \text{if } u \in V_1 \setminus \{u_1, u_2\} \\ s^2(\{u\}) - s^2(\emptyset) & \text{if } u \in \{u_1, u_2\} \\ \Delta & \text{if } u \in \{b_1, b_2\}; \end{cases}$$

$$w^2 \in \mathbb{Z}^{V(G_2^2)} \text{ by } w_u^2 := \begin{cases} w_u & \text{if } u \in V_2 \setminus \{u_1, u_2\} \\ w_u + s^2(\emptyset) - s^2(\{u\}) + \Delta & \text{if } u \in \{u_1, u_2\} \\ \Delta & \text{if } u \in \{b\}. \end{cases}$$

Claim 2:  $\alpha_{w_2}(G_1^3) = \alpha_w(G) + \Delta - s^2(\emptyset)$  and  $\alpha_{w_2}(G_2^2) = s^2(\emptyset) + \Delta$ . Moreover, for  $i=1,2$  there exists a stable set  $S$  in  $G_2^2$  with  $\sum_{u \in S} w_u^2 = \alpha_{w_2}(G_2^2)$ ,  $u_1 \notin S$ , and  $b \notin S$ .

Proof of Claim 2: Straightforward case checking.

end of proof of claim 2

By claim 1 there exists a  $w^1$ -cover  $E^1, C^1$  by edges and odd circuits  $G_1^3$  with cost  $\alpha_{w_1}(G_1^3) = \alpha_w(G) + \Delta - s^2(\emptyset)$ . Let  $\gamma_1, \gamma_2$  and  $\tilde{\gamma}$  denote the multiplicity of  $e_1, e_2, \tilde{e}$  respectively in  $E^1$ . Let  $\beta$  denote the sum of the multiplicities of the odd cycles in  $C^1$  containing  $b_1$  (and  $b_2$ ). Assume  $E^1$  and  $C^1$  are such that  $\gamma_1 + \gamma_2 + 2\tilde{\gamma} + \beta$  is minimal.

Claim 3:  $\gamma_1 + \tilde{\gamma} + \beta = \Delta$  for  $i=1,2$ . Consequently,  $\gamma_1 = \gamma_2$ .

Proof of Claim 3:  $\gamma_1 + \tilde{\gamma} + \beta \geq \Delta$  since  $E^1, C^1$  is a  $w^1$ -cover. Suppose  $\gamma_1 + \tilde{\gamma} + \beta > \Delta$ . Then  $\gamma = 0$ . Indeed, if not, then increasing  $\gamma_2$  by 1 and decreasing  $\tilde{\gamma}$  by 1 would yield a  $w^1$ -cover with cost  $\alpha_{w_1}(G_1^3)$ , and smaller  $\gamma_1 + \gamma_2 + 2\tilde{\gamma} + \beta$ . Moreover,  $\gamma_1 = 0$ . Otherwise, take some  $u_1 v \in E(G^1)$ . Adding  $u_1 v$  to  $E^1$  (or increasing its multiplicity in  $E^1$ ) and decreasing  $\gamma_1$  by 1, again yields a  $w^1$ -cover with cost  $\alpha_{w_1}(G_1^3)$ , and smaller  $\gamma_1 + \gamma_2 + 2\tilde{\gamma} + \beta$ . Finally  $\beta = 0$ , contradicting the fact that  $\Delta \geq 0$ . Indeed if  $\beta > 0$  remove an odd circuit  $C$  with  $b_1 \in V(C)$  from  $C^1$ , and add the edges in the unique maximum cardinality matching  $M \subset E(C)$  not covering  $b_1$  to  $E^1$ . Since  $M = \frac{1}{2}(|V(C)| - 1)$  this again yields a  $w^1$ -cover with cost  $\alpha_{w_1}(G_1^3)$ , and smaller  $\gamma_1 + \gamma_2 + 2\tilde{\gamma} + \beta$ .

end of proof of claim 3

By claim 1, there also exists a  $w^2$ -cover  $E^2, C^2$  by edges and odd circuits in  $G_2^2$  with cost  $\alpha_{w_2}(G_2^2) = s^2(\emptyset) + \Delta$ . Let  $E^2$  and  $C^2$  be such that the sum,  $\delta$ , of the multiplicities of the odd cycles in  $C^2$  containing  $b$  is minimal.



Claim 4:  $f_1$  and  $f_2$  do not occur (i.e. have multiplicity 0) in  $E^2$ . Moreover  $\delta = \Delta$ .

Proof of Claim 4: Since the cost of  $E^2$ ,  $C^2$  is  $\alpha_w(G_2^2)$  and there exists a stable set  $S$  in  $G_2^2$  with  $\sum_{u \in S} w_u^2 = \alpha_w(G_2^2)$  and  $u_1, b \notin S$  (Claim 2), the edge  $f_1$  does not occur in  $E^2$  ("complementary slackness"), Equivalently  $f_2$  does not occur in  $E^2$ . The proof that  $\delta = \Delta$  is similar to the proof of claim 3. end of proof of claim 4

Using  $E^1$ ,  $C^1$  and  $E^2$ ,  $C^2$  we are now able to construct a  $w$ -cover  $\tilde{E}$ ,  $\tilde{C}$  in  $G$  by edges and odd circuits, and with cost  $\alpha_w(G)$ . Thus proving  $\alpha_w(G) = \tilde{\rho}_w(G)$ . The construction goes as follows:

Step 1: The edges in  $E^1$  and  $E^2$ , except  $e_1, e_2$  and  $\tilde{e}$  are added to  $\tilde{E}$  (with the same multiplicity). The odd circuits in  $C^1$  and  $C^2$  not containing  $b_1$  ( $b_2$ ), or  $b$  are added to  $\tilde{C}$ .

Step 2: Let  $C_1^2, \dots, C_\Delta^2$  be the odd circuit in  $C^2$  containing  $b$ . (Remind that some of them may be equal.)

- (i) Let  $C_1^1, \dots, C_\beta^1$  be the odd circuits in  $C^1$  containing  $b_1$ , define for each  $i=1, \dots, \beta$  the odd circuit  $C_i \in C(G)$  by  
 $E(C_i) = E(C_i^1) \cup E(C_i^2) \setminus \{e_1, e_2, \tilde{e}, f_1, f_2\}$ . Add all the odd circuits  $C_1, \dots, C_\beta$  to  $\tilde{C}$ .  
 Note that, for each  $i=1, \dots, \beta$ :  $\frac{1}{2}|V(C_i)| - 1 = \frac{1}{2}(|V(C_i^1)| - 1) + \frac{1}{2}(|V(C_i^2)| - 1) - 2$ .

- (ii) Define for each  $i=\beta+1, \dots, \beta+\gamma_1$  the collection of edges  $M_i$  as the unique maximum cardinality matching in  $E(C_i^2)$  not covering  $b$ . Each edge occurring in  $M_i$  ( $i=\beta+1, \dots, \beta+\gamma_1$ ) is added to  $\tilde{E}$  (as often as it occurs in an  $M_i$ ).

Note that, for each  $i=\beta+1, \dots, \beta+\gamma_1$ :  $|M_i| = \frac{1}{2}(|V(C_i^2)| - 1)$ .

- (iii) Define for each  $i=\beta+\gamma_1+1, \dots, \beta+\gamma_1+\tilde{\gamma} = \Delta$  the collection of edges  $N_i$  as the unique maximum cardinality matching in  $E(C_i^2)$  not covering  $u_1$  and not covering  $u_2$ . All the edges occurring in an  $N_i$  are added to  $\tilde{E}$  (as often as they occur in an  $N_i$ ).

Note that, for each  $i=\beta+\gamma_1+1, \dots, \Delta$ ,  $|N_i| = \frac{1}{2}(|V(C_i^2)| - 1) - 1$ .



Claim 5: The collections  $\tilde{E}$ ,  $\tilde{C}$  form a  $w$ -cover by edges and odd circuits in  $G$ .

Proof of Claim 5: It is not hard to see that each  $u \in (V_1 \setminus V_2) \cup (V_2 \setminus V_1)$  is covered  $w_u$  times by  $\tilde{E}$ ,  $\tilde{C}$ . (The matchings in step 2(ii) and in step 2(iii) of the construction do not decrease the number of times that a node in  $V_2 \setminus V_1$  is covered.) The node  $u_1$  is covered at least  $s^2(\{u\}) - s^2(\emptyset)$  times by  $E^2$ ,  $C^2$ , and at least  $w_u + s^2(\emptyset) - s^2(\{u\}) + \Delta$  times by  $E^1$ ,  $C^1$ . So  $u_1$  is covered at least  $w_u + \Delta$  times by  $E^1$ ,  $C^1$  and  $E^2$ ,  $C^2$  together. During the construction this amount is decreased with  $\beta$  by step 2(i), with  $\gamma_1$  by step 2(ii), and with  $\tilde{\gamma}$  by step 2(iii). Since  $\beta + \gamma_1 + \tilde{\gamma} = \Delta$ ,  $\tilde{E}$  and  $\tilde{C}$  cover  $u_1$  at least  $w_u$  times. Similarly one deals with  $u_2$ , as  $\gamma_1 = \gamma_2$ . end of proof of claim 5

Claim 6: The cost of  $\tilde{E}$ ,  $\tilde{C}$  is  $\alpha_w(G)$ .

Proof of Claim 6: The cost of  $E^1$ ,  $C^1$  plus the cost of  $E^2$ ,  $C^2$  is equal to  $\alpha_{w_1}(G_1^3) + \alpha_{w_2}(G_2^2) = \alpha_w(G) + \Delta - s^2(\emptyset) + s^2(\emptyset) + \Delta = \alpha_w(G) + 2\Delta$ . During the construction we lost exactly:  $2\beta$  in step 2(i),  $\tilde{\gamma}$  in step 2(iii), and  $2\gamma_1 + \tilde{\gamma}$  by ignoring the edges  $e_1$ ,  $e_2$ ,  $\tilde{e}$ . So the cost of  $\tilde{E}$ ,  $\tilde{C}$  is  $\alpha_w(G) + 2\Delta - 2\beta - \tilde{\gamma} - (2\gamma_1 + \tilde{\gamma}) = \alpha_w(G)$ . end of proof of claim 6

Claim 5 and 6 together yield that  $\alpha_w(G) = \tilde{\rho}_w(G)$ .

Case IIb:  $\Delta \leq 0$ .

The proof of this case is similar to the proof of case IIa. Therefore we shall only give the beginning of it.

Let  $b$  be the new node in  $G_1^2$  and let  $b_1$  and  $b_2$  be the new nodes in  $G_2^3$  (see figure 6).

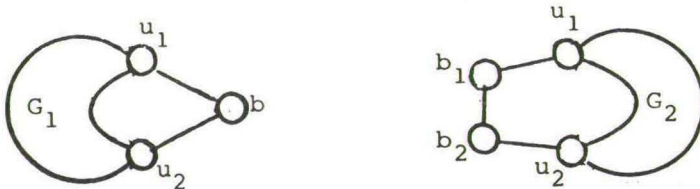


figure 6

Define the following weight functions:

$$w^1 \in \mathbb{Z}^{V(G_1^2)} \text{ by } w_u^1 := \begin{cases} w_u & \text{if } u \in V_1 \setminus V_2 \\ s^2(\{u\}) - s^2(\emptyset) - \Delta & \text{if } u \in \{u_1, u_2\} \\ -\Delta & \text{if } u = b; \end{cases}$$

$$w^2 \in \mathbb{Z}^{V(G_2^3)} \text{ by } w_u^2 := \begin{cases} w_u & \text{if } u \in V_2 \setminus V_1 \\ w_u + s^2(\emptyset) - s^2(\{u\}) & \text{if } u \in \{u_1, u_2\} \\ -\Delta & \text{if } u \in \{b_1, b_2\}. \end{cases}$$

The first thing to be proved now is

Claim 7:  $\alpha_w^1(G_1^2) = \alpha_w(G) - \Delta - s^2(\emptyset)$  and  $\alpha_w^2(G_2^3) = -\Delta + s^2(\emptyset)$ . Moreover, for each  $U \in \{\{u_1, b_1\}, \{b_1, b_2\}, \{u_2, b_2\}\}$  there exists a stable set  $S$  in  $G_2^3$  with  $\sum_{u \in S} w_u = \alpha_w^2(G_2^3)$ , and  $S \cap U = \emptyset$ .

From this point it is not hard to see how arguments similar to those used in Case IIa prove that  $\alpha_w(G) = \tilde{\rho}_w(G)$ .  $\square$

#### Remarks:

The proof of Case I of the proof above is identical with the proof of Theorem 4.1 in Chvátal [1975]. The techniques used in Case IIa and Case IIb of the proof are similar to the techniques used by Boulala and Uhry [1979]. However they restrict  $G_2$  to paths and odd cycles. Sbihi and Uhry [1984] also use the decompositions of Case II. In their case  $G_2$  is always bipartite. Recently, Barahona and Mahjoub [1986] derived a construction to derive all facets of the stable polytope of  $G$ , in case  $G$  has a two node cutset  $\{u_1, u_2\}$ , from the facets of the stable set polytopes of  $G_1^+$  and  $G_2^+$ . (Here  $G_1$  and  $G_2$  are as in the proof above,  $G_1^+$  is derived from  $G_1$  by adding a five cycle  $\{u_1, b, u_2, b_1, b_2\}$ ).

### 3. Computational Aspects

In this final section we give some attention to the computational complexity of the problems: Given  $G$  and  $w \in \mathbb{Z}^{V(G)}$ , determine  $\alpha_w(G)$ ,  $\tilde{\rho}_w(G)$ ,  $\rho_w(G)$ ,  $\tau_w(G)$ ,  $\tilde{v}_w(G)$ , and  $v_w(G)$ . Well known results are:

It is NP-hard to determine  $\alpha_w(G)$ ,  $\tau_w(G)$ , even if  $w \equiv 1$  (Karp [1972]). There exists a polynomial time algorithm to determine a maximum cardinality  $w$ -matching, or a minimum cardinality  $w$ -edge-cover (Edmonds [1965] for  $w \equiv 1$ , Cunningham and Marsh [1978] for general  $w$ ).

Pulleyblank observed that determining  $\tilde{\rho}_w(G)$ , or  $\tilde{v}_w(G)$  is NP-hard, even if  $w \equiv 1$ . There is a reduction from PARTITION INTO TRIANGLES (cf. Garey and Johnson [1979]).

Indeed, given a graph  $G$  there is partition of  $V(G)$  into triangles in  $G$  if and only if  $|\tilde{\rho}(G)| \leq \frac{1}{3}|V(G)|$ . Since PARTITION INTO TRIANGLES remains NP-complete for planar graphs (Dyer and Frieze [1986]), determining  $\tilde{\rho}(G)$ , or  $\tilde{v}(G)$  remains NP-hard even if  $G$  is planar.

If  $G$  has no odd- $K_4$   $\tilde{\rho}_w(G)$  and  $\tilde{v}_w(G)$  can be found efficiently (i.e. in polynomial time). Indeed, an algorithm can be obtained from the proofs in section 2 (proof of Theorem 2.2, proof of Theorem 1.8). The only difficulty is finding an orientation  $\vec{A}$  of discrepancy 1, and solving (2.3) and (2.4).

Finding  $\vec{A}$ : Using a constructive characterization of graphs with no odd- $K_4$  and no odd- $K_3$  (Lovasz, Schrijver, Seymour, Truemper [1984], cf. Gerards-Schrijver [1986]) similar to the result in remark (iii) of section 1, one easily derives a polynomial time algorithm to find  $\vec{A}$ , or to decide that  $\vec{A}$  does not exist (i.e. that  $G$  has an odd- $K_4$  or an odd- $K_3^2$ , Theorem 2.1).

Solving (2.3) and (2.4): Define the directed graph  $D = (V(D), A(D))$  by:  
 $V(D) := \{u_i \mid u \in V(G); i=1,2\}$ ;  $A(D) := \{\overrightarrow{u_1 u_2} \mid u \in V(G)\} \cup \{\overrightarrow{u_2 v_1} \mid \overrightarrow{uv} \in \vec{A}\}$ .  
 Then (2.3) is equivalent to the min-cost-circulation problem:

$$\begin{aligned}
 (3.2) \quad & \min \sum_{\substack{\overrightarrow{uv} \in \vec{A}}} g \overrightarrow{u_2 v_1} \\
 & \text{s.t. } g \text{ is a non-negative circulation in } D, \\
 & g \overrightarrow{u_1 u_2} \geq w_u \quad (u \in V(D)).
 \end{aligned}$$

(3.2) can be efficiently solved by the out-of-kilter method of Ford and Fulkerson [1962]. (Note that since the cost function is  $\{0,1\}$ -valued,

there is no need to appeal to more sophisticated techniques as used by Edmonds and Karp [1972], Röck [1980] or Tardos [1985].)

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